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Classical Pure States: Information and Symmetry in Statistical Mechanics

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Abstract

The aim of this paper is to find classical counterparts of pure quantum states. It is shown that these are singular probability distributions concentrated on the so-called maximal null manifolds in a phase space. They are equivalent to densities studied by Van Vleck and Schiller and to WKB solutions (cf. Van Vleck, 1928; Schiller, 1962). Properties of such distributions and their relativistic generalisations have been studied in previous papers (Sławianowski, 1971; Sławianowski, 1972). However, it has not been shown there that such distributions arise actually in the limit $h \to 0$. When working with the standard apparatus of differential geometry we mostly use the language of Kobayashi & Nomizu (1963).

1. *Introduction*

In this paper we restrict ourselves to systems of classical analogy only. It is well known that in the limit $\hbar \to 0$ quantum mechanics of such systems asymptotically approaches classical statistical mechanics. These relationships become as apparent as possible when the formulation of quantum mechanics due to Moyal, Weyl and Wigner is used (Moyal, 1949; Weyl, 1931).

Probability distributions over a phase space are classical counterparts of quantum states. It is known that pure states possess maximum information. Thus, it seems reasonable to conjecture that their classical counterparts are distributions, or measures, concentrated on submanifolds of lower dimension. One commonly believes that they are Dirac measures, i.e. their supports are single points in a phase space.

In spite of these views we are going to show that quasiclassical probability distributions corresponding to pure states are concentrated on the so-called *maximal null manifolds* (Sławianowski, 1971; Tulczyjew, unpublished lectures). Thus to any quasiclassical pure state there is attached some set of the usual, classical states—a null manifold in a phase space. Each of these classical states is taken with its own statistical weight. Thus, classical states, i.e. points in a phase space, are 'hidden parameters' of quasiclassical ones.

In previous papers (Sławianowski, 1971; Sławianowski, 1972) we quoted *formal* arguments in support of such views. One could mean that the rigorous limit-transition $\hbar \to 0$ is sufficient to make a *physical* choice between the analogies mentioned. Unfortunately it is not the case. In quantum mechanics, pure states possess many equivalent properties which could be used to define them. For example:

(i) pure states possess the maximum of information,
(ii) they are described by means of idempotent density operators,
(iii) their density operators satisfy maximal systems of independent eigenequations (in operator formulation of quantum mechanics).

All these properties possess classical counterparts which can be found by taking the limit $\hbar \to 0$. The classical pure states could be defined as probability distributions satisfying 'classical translations' of some of conditions (i), (ii) and (iii) above. Unfortunately, on the classical level, these conditions fail to be equivalent. This qualitative discontinuity of the limit-transition $\hbar \to 0$ forces us to guess the following riddle: Which of the properties (i), (ii) or (iii) have to be translated into classical language in order to obtain a satisfactory definition of classical pure states? Below, we show which conditions lead to null manifolds and which to points in a phase space. The first of them seem to be more physical. Besides, they are compatible with the quasiclassical behaviour of exact quantum Moyal–Wigner densities, which were found *before* taking the limit $\hbar \to 0$. For example, the quantum state of definite momentum P is described by the following Moyal–Wigner density function:

$$\rho_P(q^i, p_i) = \delta(p_1 - P_1) \ldots \delta(p_n - P_n)$$

where n is the number of degrees of freedom. The distribution ρ_P retains this form in the classical limit when $\hbar$ tends to zero or, equivalently, when quantum numbers tend to infinity. Thus it is valid on the classical level as well. But its support is just the maximal null manifold of definite momentum and is described by equations:

$$p_i = P_i, \qquad i = 1, 2, \ldots, n$$

The above example is rather trivial, but we show that in the classical limit all Moyal–Wigner distributions become concentrated around appropriate null manifolds. Moreover, probability distributions over null manifolds lead to densities of Van Vleck and, equivalently, WKB solutions. On the contrary, we do not know any interesting consequence of the currently used analogy between pure states and points of the phase space.

2. *Pure Quantum States*

Before studying classical counterparts we survey the usual properties of pure quantum states. Our aim in this chapter is to describe them in terms of such *physical* concepts as information and symmetry.

Let $\mathscr{A}$ be an algebra of bounded operators on a Hilbert space. Its hermitean elements correspond to physical quantities. Quantum states are described by hermitean and positively definite density operators. For simplicity we assume $\mathscr{A}$ is finite-dimensional (a system of spins, e.g.)

Pure states may be defined in two equivalent ways:

(1) Their density operators are idempotent:

$$\rho\rho = \rho \tag{2.1}$$

(2) They possess the maximum of information, i.e. their entropy does vanish:

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) = 0 \tag{2.2}$$

Both (2.1) and (2.2) are untenable for our study of classical counterparts. The purely mathematical condition (2.1) does not use any physical concepts at all. The concept of information, such as occurs in (2.2), although a physical one, is still too abstract. In our opinion it is more physical to define pure states as those giving a unique answer on the maximal number of questions (measurements).

A physical quantity $A \in \mathscr{A}$ takes a value $a \in SpA$ on the state $\rho \in \mathscr{A}$ if and only if the following operator eigenequation is satisfied:

$$A\rho = a\rho \tag{2.3}$$

Both A, ρ are hermitean; thus, combining (2.3) with its hermitean conjugate equation, one obtains:

$$\frac{1}{\hbar i}[A, \rho] = 0 \tag{2.4}$$

This means that ρ is invariant under a one-parameter unitary group generated by A:

$$\exp\left[-\frac{i}{\hbar} tA\right] \rho \exp\left[\frac{i}{\hbar} tA\right] = \rho \tag{2.5}$$

for arbitrary $t \in R^1$.

Thus, in quantum mechanics, the concepts of information and symmetry are closely related to each other. The lack of statistical dispersion of physical quantity A on the state ρ implies in the invariance of ρ under the group generated infinitesimally by A.

In general, density operators satisfying (2.3) describe mixed states. To eliminate mixed solutions of (2.3) it is necessary to add sufficient similar equations. More strictly, a density operator ρ represents a pure state if and only if it satisfies a maximal system of independent eigenequations:

$$(A_i - a_i)\rho = 0, \qquad i = 1, 2, \ldots, N \tag{2.6}$$

The word 'maximal' in the above statement means that operators $(A_i - a_i)$ generate a maximal left ideal in $\mathscr{A}$. Thus, any consistent system of eigen-equations,

$$(A_i - a_i)x = 0, \qquad Fx = 0$$

is equivalent to the first of them alone,

$$(A_i - a_i)x = 0$$

Hence there exist operators $F_i \in \mathscr{A}$, $i = 1, 2, \ldots, N$, such that:

$$F = F^i(A_i - a_i)$$

The accuracy of the above description of pure states follows easily from the properties of the canonical basis in H^+-algebra (Ambrose, 1945; Loomis, 1953).

Let us observe the following consequences of the maximal system (2.6):

$$\frac{1}{\hbar i}[A_i, \rho] = 0 \tag{2.7}$$

$$\frac{1}{\hbar i}[A_i, A_j] = C^k_{ij}(A_k - a_k) \tag{2.8}$$

where C^k_{ij} are arbitrary operators (Dirac, 1964).

All the above statements may be summarised as follows:

Proposition 1

(i) A density operator ρ describes a pure state if and only if the subset $E_\rho = \{F \in \mathscr{A} : F\rho = 0\}$ is a maximal left ideal in $\mathscr{A}$.

(ii) A real linear subspace of E_ρ composed of hermitean operators is a Lie subalgebra of $\mathscr{A}$ in the sense of the quantum Poisson bracket

$$\frac{1}{\hbar i}[F, G] \in E_\rho$$

provided $F, G \in E_\rho$ and both F, G are hermitean.

(iii) ρ is an invariant of Lie subalgebra E_ρ, i.e.

$$\frac{1}{\hbar i}[F, \rho] = 0$$

provided F is a hermitean element of E_ρ.

In conclusion, we note that the pure states possess two fundamental *physical* properties: maximum *information* and *symmetry*. The second property is implied by the first one. Let us briefly summarise them:

(A) *Maximum information:* ρ describes a pure state if and only if the maximal system of eigenequations (2.6) is satisfied:

$$A_i \rho = a_i \rho, \qquad i = 1, 2, \ldots, N$$

The operators $(A_i - a_i)$ appearing in (2.6) generate the *maximal* left ideal E_ρ in $\mathscr{A}$.

(B) *The symmetry:* ρ is invariant under all one-parameter unitary subgroups generated by hermitean elements of E_ρ (for example by A_i).

The currently used definition of pure states is based on (2.1) $\rho\rho = \rho$. It appears, however, that the description based on Proposition 1, although equivalent to (2.1), is much more physical. This point is of great importance in the study of classical counterparts. On the classical level, an equivalence of the approaches mentioned breaks down. Thus the more physical one has to be chosen.

3. *The Weyl Prescription and Classical Limit*

Let (P, γ) be a phase space, i.e. a smooth differential manifold P equipped with a non-degenerate and closed two-form γ (Sławianowski, 1971). We also assume the existence of an affine structure in P. This means that P is a homogeneous space of a linear space of translations Π. All translations are assumed to be canonical mappings in P, i.e. they preserve γ. A natural symplectic two-form induced by γ on the space Π will be denoted by Γ. The only translation which maps $p_1 \in P$ onto $p_2 \in P$ is denoted by $\overrightarrow{p_1 p_2} \in \Pi$.

The formulation of quantum mechanics due to Moyal, Weyl and Wigner provides a convenient framework for studying quasiclassical problems. The reason is that this approach is formally analogous to classical statistical mechanics. For example, it makes use of the same classical phase space (P, γ). Physical quantities and statistical states (ensembles) are represented in both theories by real functions over P. The difference lies in algebraic structures which are surmised in the underlying set of functions. In classical algebra, the ordinary commutative multiplication of functions is used. The quantum theory is based on the non-local, non-commutative Weyl product. It is given by:

$$(A \,\square\, B)(p) = \left(\frac{2}{2\pi\hbar}\right)^{2n} \int \exp \frac{2i}{\hbar} \langle \Gamma, \overrightarrow{p_1 p} \wedge \overrightarrow{p_2 p} \rangle A(p_1) B(p_2)\, dp_1\, dp_2 \tag{3.1}$$

where dp_i is an invariant measure on P.

Both classical and quantum algebras make use of the same, natural hermitean involution and scalar product:

$$A^+ = A^*, \qquad \langle A | B \rangle = \int A^*(p) B(p)\, dp \tag{3.2}$$

Thus, expectation values and probabilities of results of measurements are given by the same formulas in both theories:

$$\langle A \rangle_\rho = \langle A | \rho \rangle, \qquad P(\rho_1, \rho_2) = \langle \rho_1 | \rho_2 \rangle$$

($P(\rho_1, \rho_2)$ is the probability of finding the system in a statistical state, ρ_1, when it is known to be in a state ρ_2). The classical non-negative definiteness of probability distributions $\langle \rho | A^* A \rangle \geqslant 0$ (for all functions of A), is replaced in quantum theory by the following, similar property, $\langle\rho|A^* \square A\rangle \geqslant 0$ (for all functions of A).

There exists an isomorphism, the so-called Weyl prescription, which maps the quantum algebra of functions on P onto the usual algebra of operators, $\mathscr{A}$. The inverse of the Weyl prescription carries over the full structure of $\mathscr{A}$ into the set of functions on P. For example, the complex conjugation of any function on P, and its integral over the phase space, correspond to the hermitean conjugation and to the trace of operators, respectively. The product of operators is represented by the Weyl product (3.1), and the quantum Poisson bracket by the following Moyal bracket:

$$\frac{1}{\hbar i}(A \square B - B \square A) \tag{3.3}$$

In the limit $\hbar \to 0$ the Weyl product of any smooth functions asymptotically approaches their usual, pointwise product. Similarly, the Moyal bracket tends to the classical Poisson bracket:

$$A \square B \approx AB, \qquad \frac{1}{\hbar i}(A \square B - B \square A) \approx \{A, B\}, \qquad \text{when } \hbar \to 0 \tag{3.4}$$

In such a way, in the limit $\hbar \to 0$, quantum mechanics becomes asymptotically equivalent to classical statistical mechanics. For example, the non-local and non-deterministic quantum Liouville equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{\hbar i}(H \square \rho - \rho \square H)$$

is then replaced by the classical one:

$$\frac{\partial \rho}{\partial t} = \{H, \rho\}$$

4. *Classical Pure States*

In what follows, the Moyal–Wigner formulation of quantum mechanics is consequently used.

If ρ describes an eigenstate of physical quantity A, then (2.3) and (2.4) imply, via the Weyl prescription,

$$A \square \rho = a\rho \tag{4.1}$$

$$\frac{1}{\hbar i}(A \square \rho - \rho \square A) = 0 \tag{4.2}$$

where a is an eigenvalue of the operator corresponding to A in the sense of the Weyl prescription.

The conditions above give an account of the two fundamental properties of eigenstates: the vanishing of statistical dispersion and symmetry. When $h \to 0$, these conditions become

$$A \,.\, \rho = a\rho \tag{4.3}$$

$$\{A, \rho\} = 0 \tag{4.4}$$

where $a \in A(P)$.

It appears that physical interpretation of classical conditions (4.3) and (4.4) is the same as that of the quantum ones, (4.1) and (4.2). In fact (4.3) is equivalent to the *vanishing of dispersion* of A on the classical statistical ensemble ρ. The function A takes a constant value a on the support of ρ. Therefore, one does not observe any statistical spread of results of measurements. Equation (4.4) means that ρ is *invariant* under a one-parameter group of canonical transformations generated by A. Thus, the physical interpretation of equations (4.1) and (4.2) in terms of information and symmetry does not change in the limit $h \to 0$. On the contrary, the logical relationship of information and symmetry changes abruptly. Although (4.2) follows from (4.1) the corresponding classical equations (4.3) and (4.4) are independent. On the classical level, the correlation of information and symmetry breaks down. This phenomenon gives rise to unpleasant arbitrariness in the choice of classical counterparts of eigenstates. In quantum theory, eigenstates could be defined by means of (4.1) only; (4.2) was then satisfied automatically. *A priori*, it is not clear if, on the classical level, the definition of eigenstates has to use (4.3) only, or both (4.3) and (4.4). Physical reasons rather suggest the second possibility; (4.4) describes the physical property of *symmetry*, as does (4.2). Rejecting this equation one obtains a strange result that eigenstates lose their fundamental physical symmetries in the classical limit. It is hard to accept this result. Thus, both formal and *physical* agreement with quantum concepts is attained when classical eigenstates are defined by means of both (4.3) and (4.4).

Proposition 2

A probability distribution ρ over a phase space P is a classical eigenstate of physical quantity A with an eigenvalue $a \in A(P)$ when the system of equations (4.3) and (4.4) is satisfied:

$$A \,.\, \rho = a\rho, \qquad \{A, \rho\} = 0$$

Equation (4.3) means that ρ vanishes beyond the subset

$$M_a = \{p \in P \colon A(p) = a\}$$

Let us assume that the differential dA does not vanish identically in any open submanifold and, consequently, almost all M_a are hypersurfaces. This is true for all physical quantities of practical importance. Then (4.3) does not possess any physical solution in the class of ordinary functions. The only solutions satisfying physical requirements of non-negative definiteness and normalisation are distributions

$$\rho = F\delta(A-a) \tag{4.5}$$

When ρ, in addition, satisfies (4.4) then:

$$\{F, A\}|M_a = 0$$

Proposition 3

The classical eigenstates of physical quantity A, with an eigenvalue $a \in A(P)$, are given by:

$$\rho = F\delta(A-a), \qquad \text{where } \{F, A\}|M_a = 0 \tag{4.6}$$

(Assuming, A is not constant in any open subset, of course.)

Remark

Instead of distributions, the differential forms on hypersurfaces could be used. Let Θ be an arbitrary $(2n-1)$ form satisfying

$$\gamma^n = \gamma \wedge \ldots \wedge \gamma = dA \wedge \Theta \tag{4.7}$$

The form $\Theta_{(A,a)} = \Theta|M_a$ does not depend on any particular choice of Θ satisfying (4.7), thus it is characterised uniquely by A and a. It defines some probabilistic measure μ_a on M_a. This measure is equal to that induced by $\delta(A-a)$, i.e. for any open subset V of M_a we have:

$$\mu_a(V) = \int_V \Theta_{(A,a)} = \int_{\bar{V}} \delta(A-a)\gamma^n = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{\bar{V}} \exp[ik(A-a)]\gamma^n$$

where $\bar{V}$ is an arbitrary open subset of P satisfying $\bar{V} \cap M_a = V$.

In a close analogy to quantum theory, classical pure states are defined as those satisfying maximal systems of 'eigenconditions'

$$A_i \cdot \rho = a_i \rho \tag{4.8}$$

$$\{A_i, \rho\} = 0, \qquad i = 1, 2, \ldots, N \tag{4.9}$$

'Maximal' means here that any extended system of eigenconditions

$$A_i . x = a_i x \qquad F.x = 0$$
$$\{A_i, x\} = 0 \qquad \{F, x\} = 0$$

is consistent if and only if $F = F^j.(A_j - a_j)$, where F^j are some smooth functions on P.

The classical properties of the Poisson bracket (Caratheodory, 1956; Whittaker, 1952; Sławianowski, 1971) imply that

$$\{A_i - a_i, A_k - a_k\} = \{A_i, A_k\} = C^l_{ik}.(A_l - a_l) \tag{4.10}$$
$$N = n = \tfrac{1}{2}\dim P$$

Thus the word 'maximal' now becomes exactly defined. It should be noticed, however, that our treatment is not very rigorous: we disregard degeneracy and all related global problems in P.

The most general probability distribution satisfying (4.8) and (4.9) is given by:

$$\rho = \Phi\,\delta(A_1 - a_1)\ldots\delta(A_n - a_n)$$

where

$$\{\Phi, A_i\}|\mathcal{M}_a = 0, \qquad i = 1, 2, \ldots, n,$$

and

$$\mathcal{M}_a = \{p \in P: A_j(p) = a_j, \qquad j = 1, 2, \ldots, n\}$$

Hence it is enough to put $\Phi = F(A_1 \ldots A_n)$ in order to obtain the most general distribution ρ (F is any smooth function).

Proposition 4

A probability distribution ρ defined over $2n$-dimensional phase space is a classical counterpart of pure quantum state if and only if there exist n independent functions $A_1 \ldots A_n$ such that

$$\rho = F(A_1 \ldots A_n)\,\delta(A_1)\ldots\delta(A_n) \tag{4.11}$$

and

$$\{A_i, A_j\} = C^k_{ij}.A_k \tag{4.12}$$

where F and C^k_{ij} are some smooth functions on R^n and P respectively.

Now, let us consider the following submanifold in a phase space P:

$$\mathcal{M} = \{p \in P: A_i(p) = 0, \qquad i = 1, 2, \ldots, n\}$$

where $A_1, \ldots, A_n$ are functionally independent functions satisfying equations (4.12). Those equations express the fact that the two-form γ vanishes when restricted to submanifold $\mathcal{M}$. Hence (4.12) may be written down as follows:

$$\gamma|\mathcal{M} = 0$$

(Sławianowski, 1971). Thus $\mathcal{M}$ is a null submanifold of maximal dimension in P. Any two vectors u, v, tangent to $\mathcal{M}$ and attached at the same point $p \in \mathcal{M}$, are γ-orthogonal, i.e.

$$\langle \gamma_p, u \wedge v \rangle = 0$$

The property above gives an account of the geometric structure of $\mathcal{M}$. It appears that null submanifolds, independent of their physical interpretation, present interest for the pure mathematicians as well.

The notion of null manifolds enables us to formulate the main result of our investigation as follows:

Proposition 5

Classical pure states are probability distributions concentrated on maximal null submanifolds in a phase space.

It is not difficult to find a classical counterpart of Proposition 1:

Proposition 6

Let ρ be a probability distribution and $\mathscr{E}_\rho$ an ideal in the associative algebra of smooth functions on P, defined as follows:

$$\mathscr{E}_\rho = \{F \in C^\infty(P) : F.\rho = 0\}$$

Then ρ is a classical pure state if and only if $\mathscr{E}_\rho$ is a maximal ideal being at the same time a Lie subalgebra of $C^\infty(P)$ in the sense of a Poisson bracket:

$$\{F_1, F_2\} \in \mathscr{E}_\rho, \qquad \text{provided } F_1, F_2 \in \mathscr{E}_\rho$$

Obviously ρ is an invariant of $\mathscr{E}_\rho$,

Remark

The word '*maximal*' in propositions 5, 6 is used in the sense of dimension and functional independence respectively. Nevertheless, we suspect, it could be used literally, if we replaced $C^\infty(P)$ by a class of *analytic* functions $C^\omega(P)$ and smooth submanifolds $\mathscr{M}$ by analytical ones. Obviously, P has to be assumed then to be an analytic manifold.

The difference between Propositions 1 and 6 lies in the logical relationship between the structure of associative algebra and that of Lie algebra. In the classical limit those structures become separated. The quantum Poisson bracket is algebraically built of the associative Weyl product (3.1); on the other hand, the classical Poisson bracket and ordinary pointwise product are algebraically independent. This reflects the aforementioned independence of information and symmetry on the classical level.

Neglecting (4.4) and using (4.3) only in the definition of classical eigenstates, one would obtain distributions concentrated on single points. In fact, the maximal number of independent 'eigenconditions' could be equal $2n$. The current analogy between pure states and points of the phase space overlooks the fundamental symmetry condition (4.4).

The question arises if the idempotence condition (2.1) does not enable us to choose uniquely one of the mentioned analogies. However, it is not the case; (2.1) is compatible with both of them.

5. *Null Manifolds and the WKB Approximation*

Let us consider a mechanical system whose configuration space is an affine manifold Q. Affine coordinates in Q will be denoted by $q^1, \ldots, q^n$. They induce, in a natural way, canonical coordinates $\bar{q}^1, \ldots, \bar{q}^n, p_1, \ldots, p_n$ in a phase space $P = T^*Q$; obviously, $\gamma = dp_i \wedge d\bar{q}^i$ (Sławianowski, 1971). To make our notation less complicated we do not distinguish below between q^i and $\bar{q}^i$, although they are different functions defined on different manifolds ($\bar{q}^i = q^i \circ \tau^*$, where $\tau^* : T^*Q \to Q$ is the projection of the cotangent bundle onto its base). In what follows this simplification does not lead to misunderstandings.

Let P be foliated by a family of orientable null manifolds:

$$\mathscr{M}_a = \{p \in P : A_i(p) = a_i, \qquad i = 1, 2, \ldots, n\} \tag{5.1}$$

where a is a shorthand notation for $(a_1, \ldots, a_n)$ and

$$\{A_i, A_j\} = 0$$

Besides, we assume that all $\mathscr{M}_a$ are images of cross-sections

$$\sigma_a : Q \to P = T^* Q$$

(Sławianowski, 1971). Hence, equations (5.1) may be transformed into

$$\left[p_i - \frac{\partial S}{\partial q^i}(q, a) \right] \Big| \mathscr{M}_a = 0 \tag{5.2}$$

The distribution $\delta(A_1 - a_1) \ldots \delta(A_n - a_n)$ defines some measure on the manifold $\mathscr{M}_a$. This measure admits an alternative description in terms of some differential n-form Θ_a on $\mathscr{M}_a$. This form is constructed similarly to that in the previous chapter. Obviously, Θ_a is non-negative with respect to one of the two admissible orientations on $\mathscr{M}_a$.

The cross-section $\sigma_a : Q \to P$ enables us to project Θ_a to Q:

$$\mathscr{V}_a = \sigma_a{}^* . \Theta_a$$

The form $\mathscr{V}_a$ is equivalent to Van Vleck density (Van Vleck, 1928; Schiller, 1962; Sławianowski, 1972),

$$\mathscr{V}_a = \det \left\| \frac{\partial^2 S}{\partial q^i \, \partial a^j}(., a) \right\| dq^1 \wedge \ldots \wedge dq^n \tag{5.3}$$

up to a non-essential constant factor (Sławianowski, 1972).

Let us now build the following wave functions on Q:

$$\Psi_a = \sqrt{\left(\det \left\| \frac{\partial^2 S}{\partial q^i \, \partial a^j}(., a) \right\| \right)} \exp \frac{i}{\hbar} S(., a) \tag{5.4}$$

They are easily recognised as WKB solutions of eigenequations

$$\hat{A}_i \Psi_a = a_i \Psi_a, \qquad i = 1, 2, \ldots, n \tag{5.5}$$

where operators $\hat{A}_i$ correspond to functions A_i via the Weyl prescription. Thus our concept of classical pure states leads to densities discovered by Van Vleck in 1928, which are equivalent to WKB solutions.

Let us now notice another interesting connection between null manifolds and wave functions. The Moyal–Wigner density corresponding to wave function Ψ is given by

$$\rho(q^i, p_i) = \left(\frac{1}{2\pi} \right)^n \int \Psi^* \left(q^i - \frac{\hbar \tau^i}{2} \right) \exp(-i \tau^i p_i) \, \Psi \left(q^i + \frac{\hbar \tau^i}{2} \right) d_n \tau \tag{5.6}$$

(Moyal, 1949). (For brevity we do not distinguish between functions on manifolds and their expressions in coordinates). Let us put

$$\Psi = \sqrt{(D)} \exp \frac{i}{\hbar} S$$

One supposes (in the WKB approximation) that both D, S do not depend on $\hbar$. The independence of their physical interpretation of the Planck constant is obvious even without appealing to WKB: the function D is a probability distribution for positions, S describes a spread of momentum in the Ψ-ensemble:

$$\langle\Psi|\hat{P}_i|\Psi\rangle = \int D(q)\frac{\partial S}{\partial q^i} d_n q \tag{5.7}$$

where $d_n q$ is a translationally invariant measure on Q. Hence the physical interpretation of functions D, S is evidently independent of $\hbar$.

Making use of the aforementioned independence of functions D, S of $\hbar$ one finds, in the limit $\hbar \to 0$,

$$\rho_0(q^i, p_i) = \lim_{\hbar\to 0} \rho(q^i p_i) = |\Psi(q)|^2 \delta\left(p_1 - \frac{\partial S}{\partial q^1}\right) \ldots \delta\left(p_n - \frac{\partial S}{\partial q^n}\right) \tag{5.8}$$

This means that ρ_0 is concentrated on the maximal null manifold $\mathscr{M}_S$, given by equations

$$\left(p_i - \frac{\partial S}{\partial q^i}\right)\Big|\mathscr{M}_S = 0, \qquad i = 1, 2, \ldots, n \tag{5.9}$$

('Quantum' properties of such manifolds have been studied by Dirac and Synge.) The point with coordinates $(q^i, (\partial S/\partial q^i))$ belongs to $\mathscr{M}_S$ with the weight $D(q) = \Psi^*_{(q)} \Psi_{(q)}$.

The Moyal–Wigner distributions corresponding to states of definite momentum, or position, are concentrated on the appropriate null manifolds on both the classical and quantum level.

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